Bi-polynomial rank and determinantal complexity

Akihiro Yabe

Graduate School of Information Science and Technology
The University of Tokyo
akihiro_yabe@mist.i.u-tokyo.ac.jp

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Abstract

The permanent vs. determinant problem is one of the most important problems in theoretical computer science, and is the main target of geometric complexity theory proposed by Mulmuley and Sohoni. The current best lower bound for the determinantal complexity of the d by d permanent polynomial is $d^2/2$, due to Mignon and Ressayre in 2004. Inspired by their proof method, we introduce a natural rank concept of polynomials, called the bi-polynomial rank. The bi-polynomial rank is related to width of an arithmetic branching program. We prove that the bi-polynomial rank gives a lower bound of the determinantal complexity. As a consequence, the above Mignon and Ressayre bound is improved to $(d-1)^2+1$ over the field of reals. We show that the computation of the bi-polynomial rank is formulated as a rank minimization problem. We propose a computational approach for giving a lower bound of this rank minimization, via techniques of the concave minimization. This also yields a new strategy to attack the permanent vs. determinant problem.

1 Introduction

The determinant $\det(A)$ and the permanent $\mathrm{perm}(A)$ of a square matrix $A = (a_{i,j})$ of size d are defined by

$$\det(A) := \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sign}(\sigma) \prod_{i=1}^{d} a_{i,\sigma(i)},$$

$$\mathrm{perm}(A) := \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{d} a_{i,\sigma(i)},$$

where $\mathfrak{S}_d$ is the set of permutations on $\{1, 2, \ldots, d\}$. Determinant is a representative function which admits efficient computation only with arithmetic operations. On the other hand, such an efficient computation for permanent is not known. Valiant [17] proved that the computation of permanent of 0-1 matrices is #**P**-complete. Therefore, in contrast to determinant, it is conjectured that permanent cannot be computed in polynomial time.

The determinantal complexity is a measure for the difficulty of evaluation of polynomials. Let K be a field, and let $K[x] = K[x_1, x_2, \ldots, x_D]$ denote the set of polynomials of variables $x_1, x_2, \ldots, x_D$ with coefficients in K. By an *affine polynomial matrix*, we mean a matrix each of whose entries is an affine polynomial (a linear polynomial including a constant term).

Definition 1.1 (see [10]). The *determinantal complexity* $\mathrm{dc}(p)$ of $p \in K[x]$ is defined as the minimum number n such that there exists an affine polynomial matrix $Q \in (K[x])^{n\times n}$ satisfying

$$p = \det(Q).$$

It is known in [16, 19] that if a polynomial p can be evaluated with number m of arithmetics, then the determinantal complexity $\mathrm{dc}(p)$ is $O(m^{c \log m})$ for some constant c.

Permanent is regarded as a polynomial of matrix entries. Let perm_d denote the permanent polynomial for $D = d \times d$ variables of matrix entries. If $\mathrm{dc}(\mathrm{perm}_d) = d^{\omega(\log d)}$ over K, then permanent cannot be computed by polynomial number of arithmetics on K. The following is one of the main conjectures in algebraic complexity theory (see [2]).

Conjecture 1.2. *Over a field K of characteristic not equal to two, it holds that*

$$\mathrm{dc}(\mathrm{perm}_d) = d^{\omega(\log d)}.$$

This conjecture implies $\mathbf{VP}_K \neq \mathbf{VNP}_K$, an arithmetic counterpart of $\mathbf{P}$ vs. $\mathbf{NP}$ conjecture, since permanent is in $\mathbf{VNP}_K$ (in fact $\mathbf{VNP}_K$-complete) if the characteristic of K is not equal to two [16].

The current best lower bound for $\mathrm{dc}(\mathrm{perm}_d)$, due to Mignon and Ressayre [10], is quadratic.

Theorem 1.3 (Mignon and Ressayre [10]). *Over a field K of characteristic zero, it holds that*

$$\mathrm{dc}(\mathrm{perm}_d) \geq \frac{d^2}{2}.$$

Improving this bound is one of the most prominent issues in the literature. Cai, Chen and Li [3] proved that $\mathrm{dc}(\mathrm{perm}_d) \geq (d-2)(d-3)/2$ over any field K of characteristic not equal to two. Mulmuley and Sohoni [11] proposed a magnificent program, called *geometric complexity theory* (GCT), to obtain super-polynomial lower bounds by utilizing deep techniques of algebraic geometry and representation theory (also, see [6, Chapter 13]). In the context of GCT, Landsberg, Manivel and Ressayre [7] proved that the same lower bound $d^2/2$ holds for the orbit closure version $\overline{\mathrm{dc}}$ of the determinantal complexity.

Our contribution. We introduce the bi-polynomial rank of a homogeneous polynomial of even degree, and prove that the determinantal complexity is bounded below by the bi-polynomial rank. Our technique may be viewed as a higher order generalization of the Hessian rank comparison proof of the above $d^2/2$ bound (Theorem 1.3) by Mignon and Ressayre. Let $K[x]^{(k)} \subseteq K[x]$ denote the set of homogeneous polynomials of degree k.

Definition 1.4. The *bi-polynomial rank* $\mathrm{b\text{-}rank}(p)$ of $p \in K[x]^{(2k)}$ is defined as the minimum number n such that there exist $2n$ polynomials $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n \in K[x]^{(k)}$ satisfying

$$p = \sum_{i=1}^{n} f_i g_i.$$

For $p \in K[x]$ and $x_0 \in K^D$, we define a polynomial p_{x_0} by $p_{x_0}(x) := p(x + x_0)$. We denote by $p_{x_0}^{(k)}$ the degree-k homogeneous part of p_{x_0}. The set of points $z \in K^D$ with $p(z) = 0$ is denoted by $\mathrm{Zeros}(p)$. Our main result is the following.

Theorem 1.5. *For a polynomial $p \in K[x]$, $k \in [1, D]$, and $x_0 \in \mathrm{Zeros}(p)$, it holds that*

$$\mathrm{dc}(p) \geq \frac{1}{2^{2k-2}} \, \text{b-rank}(p_{x_0}^{(2k)}) - 2(k-1)D^{k-1}.$$

We will see in Section 2.2 that every generic polynomial $p \in K[x]^{(2k)}$ has the bi-polynomial rank at least $k!D^k/2(2k)!$. This means that the bi-polynomial rank has a potential to give $\Omega((d^2)^k)$ lower bound to $\mathrm{dc}(\mathrm{perm}_d)$ for every k. A direct implication to the permanent vs. determinant problem is the following.

Corollary 1.6. *Let $k \geq 1$ be an arbitrary integer. If there exists a sequence of matrices $X_d \in \mathrm{Zeros}(\mathrm{perm}_d)$ for $d = 1, 2, \ldots$ such that* $\text{b-rank}(\mathrm{perm}_{d,X_d}^{(2k)}) = \Omega(d^{2k})$, *then* $\mathrm{dc}(\mathrm{perm}_d) = \Omega(d^{2k})$.

In the case $k = 1$, our approach sharpens the Hessian approach by Mignon and Ressayre. We will see in Section 3.1 that Theorem 1.5 directly implies Theorem 1.3. Furthermore, over the field $\mathbf{R}$, our approach improves the quadratic bound as follows.

Theorem 1.7. *Over the field $\mathbf{R}$, it holds that*

$$\mathrm{dc}(\mathrm{perm}_d) \geq (d-1)^2 + 1.$$

Bounding b-rank via concave minimization. In the case $k \geq 2$, the direct calculation of the bi-polynomial rank is still difficult. We propose the following computational procedure to bound the bi-polynomial rank over $\mathbf{R}$. Let Sym_n and Psd_n denote the sets of real symmetric and positive semidefinite matrices of size n, respectively. Suppose $p \in \mathbf{R}[x]^{(2k)}$. We will show that b-rank(p) is at least the half of the minimum rank of a matrix of size $n = 2s_k$ in $\mathcal{X}_p \cap \mathrm{Psd}_n$, where $s_k := \binom{D+k-1}{k}$ and $\mathcal{X}_p$ is an affine subspace in Sym_n explicitly represented by linear equations determined by the coefficients of p; see the concrete definition in Section 2.1. Thus b-rank(p) is more than $r/2$ if the sum $\mu_{n-r}(X)$ of the smallest $n-r$ eigenvalues of X is positive for all $X \in \mathcal{X}_p \cap \mathrm{Psd}_n$. It is known that the function $X \mapsto \mu_{n-r}(X)$ is concave on Sym_n. A well-known fact in concave function minimization theory [13] tells us that if we know a polyhedral convex set $\mathcal{P} \subseteq \mathrm{Sym}_n$ containing $\mathcal{X}_p \cap \mathrm{Psd}_n$, then the minimum of μ_{n-r} is attained by extreme points of $\mathcal{P}$. Thus, the positivity of μ_{n-r} for all these extreme points is a certificate of b-rank$(p) \geq r/2$.

Proposition 1.8. *Let $p \in \mathbf{R}[x]^{(2k)}$, $r \in \mathbf{N}$, and $n = 2s_k$. If there exists $\mathcal{Y} \subseteq \mathrm{Sym}_n$ satisfying the following property, then* b-rank$(p) > r/2$.
(i) $\mathcal{X}_p \cap \mathrm{Psd}_n \subseteq \mathrm{conv}(\mathcal{Y})$.
(ii) $\mu_{n-r}(Y) > 0$ *for all* $Y \in \mathcal{Y}$.

It should be noted that this approach is essentially an outer approximation algorithm [5, 15] in the concave minimization.

Related work. The bi-polynomial rank b-rank$(p_{x_0}^{(2k)})$ can be interpreted as the minimum width of the kth layer of an arithmetic branching program (ABP) computing $p_{x_0}^{(2k)}$. Since the determinant polynomial of a matrix of size n has an ABP with width at most n^2 [8], it directly follows that a simple but weaker bound $\mathrm{dc}(p) \geq \sqrt{\text{b-rank}(p_{x_0}^{(2k)})}$. We include the detailed discussion in Section 2.3. Our bound shows the possibility to prove $\mathrm{dc}(\mathrm{perm}_d) = \Omega(d^4)$ by considering forth derivatives $\mathrm{perm}_{d,X_d}^{(4)}$ of perm_d, which seems significantly simpler than considering eighth derivatives.

Our proof method of Theorem 1.5 is first considering a normal form of an affine polynomial matrix Q, and then constructing an ABP of $\det(Q)^{(2k)}$ with small width using an exhaustive construction of low-degree terms. Such an exhaustive construction implicitly appears in the area of depth reduction of arithmetic circuits [19].

Nisan [12] considered the rank of a matrix defined by partial derivatives of non-commutative determinant, and proved an exponential lower bound of the size of ABP of non-commutative determinant. This implies an exponential lower bound of the size of non-commutative formulas for determinant. We consider the bi-polynomial rank, which is width of ABP, and formulate the bi-polynomial rank as the minimum matrix-rank over an affine subset of matrices. Therefore our approach may be viewed as a commutative analogue of Nisan's approach.

The difficulty of lower bound problems come from that of proving non-existence of certain objects. The essential idea in GCT [11] is to flip the non-existence of embeddings into the existence of representation-theoretical obstructions. Our approach might yield a comparable optimization-theoretic flip strategy for the permanent vs. determinant problem: for proving $\mathrm{dc}(\mathrm{perm}_d) = \Omega(d^{2k})$,

> find $X_d \in \mathrm{Zeros}(\mathrm{perm}_d)$, a polyhedron $\mathcal{P}_d$ containing $\mathcal{X}_p \cap \mathrm{Psd}_n$ for $p = \mathrm{perm}_{d,X_d}^{(2k)}$, and $r = O(d^{2k})$ such that $\mu_{n-r}(P) > 0$ holds for all extreme points P of $\mathcal{P}_d$.

Though much still remains to be unsettled, we hope that our approach will bring a new inspiration and trigger a new attack to this extremely difficult lower bound issue.

Organization. In Section 2, we prove basic properties of the bi-polynomial rank. In Section 2.1, we introduce a formulation of the bi-polynomial rank as the minimum matrix-rank over an affine subspace of matrices. In Section 2.2, we prove that generic polynomials $p \in K[x]^{(2k)}$ have the bi-polynomial rank at least $D^k/(2k)!$. In Section 2.3, we discuss a relation between the bi-polynomial rank and ABP. In Section 3, we consider lower bounds of $\mathrm{dc}(\mathrm{perm}_d)$ from the bi-polynomial rank for the case $k = 1$. In Section 3.1, we demonstrate that the bi-polynomial rank generalizes the Hessian rank, and give an alternative and conceptually simpler proof of Theorem 1.3. In Section 3.2, we prove $\mathrm{dc}(\mathrm{perm}_d) \geq (d-1)^2 + 1$ over the real field (Theorem 1.7). In Section 4, we prove Proposition 1.8, and propose an approach for the permanent vs. determinant problem based on the bi-polynomial rank, the rank minimization, and the concave minimization for $k \geq 2$. In Section 5, we prove Theorem 1.5, the main result of this paper.

2 Basic properties of b-rank

2.1 Rank minimization for b-rank

To consider the calculation of the bi-polynomial rank, we formulate the bi-polynomial rank as the minimum matrix-rank over an affine subspace of matrices. This formulation is a basis for discussions in subsequent sections.

Let $\mathrm{Mat}_n(K)$ be the set of square matrices of size n over the field K. For a nonnegative integer k, let $\mathcal{I}_k (= \mathcal{I}_{k,D})$ denote the set of D-tuples $(i_1, i_2, \dots, i_D)$ of nonnegative integers such that the sum $i_1 + i_2 + \cdots + i_D$ is equal to k. For $I = (i_1, i_2, \dots, i_D) \in \mathcal{I}_k$, let x^I denote the monomial $x_1^{i_1} x_2^{i_2} \cdots x_D^{i_D}$. We define $s_k (= s_{k,D}) := |\mathcal{I}_k| = \binom{D+k-1}{k}$, and consider that s_k-dimensional vectors $\boldsymbol{u} = (u_I)_{I \in \mathcal{I}_k}$ and matrices $Q = (q_{I,J})_{I,J \in \mathcal{I}_k}$ of size s_k are indexed by elements of $\mathcal{I}_k$. Then their products are written as $(Q\boldsymbol{u})_I = \sum_{J \in \mathcal{I}_k} q_{I,J} u_J$. Let $\boldsymbol{v}(x) := (x^I)_{I \in \mathcal{I}_k}$ be the s_k-dimensional vector which consists of monomials.

Theorem 2.1. *For $p \in K[x]^{(2k)}$,* b-rank(p) *is equal to the optimum value of the following problem:*

$$\begin{array}{ll} \text{Minimize} & \operatorname{rank}(Q) \\ \text{subject to} & p(x) = \boldsymbol{v}(x)^\top Q \boldsymbol{v}(x), \\ & Q \in \operatorname{Mat}_{s_k}(K). \end{array}$$

Proof. Suppose that Q_{opt} attains the optimum value. First we prove that b-rank$(p) \leq$ rank(Q_{opt}). Let $r :=$ rank(Q_{opt}). We can represent Q_{opt} as a sum $Q_{\text{opt}} = \sum_{i=1}^r \boldsymbol{f}_i \boldsymbol{g}_i^\top$ of rank one matrices $\boldsymbol{f}_i \boldsymbol{g}_i^\top$, where $\boldsymbol{f}_i$ and $\boldsymbol{g}_i$ are s_k-dimensional vectors for $i = 1, 2, \ldots, r$. Then we have

$$p(x) = \boldsymbol{v}(x)^\top Q_{\text{opt}} \boldsymbol{v}(x) = \sum_{i=1}^r (\boldsymbol{f}_i^\top \boldsymbol{v}(x))(\boldsymbol{g}_i^\top \boldsymbol{v}(x)).$$

Choosing $2r$ polynomials $f_1, f_2, \ldots, f_r, g_1, g_2, \ldots, g_r \in K[x]^{(k)}$ as $f_i(x) = \boldsymbol{f}_i^\top \boldsymbol{v}(x)$ and $g_i(x) = \boldsymbol{g}_i^\top \boldsymbol{v}(x)$ for $i = 1, 2, \ldots, r$, we have b-rank$(p) \leq r =$ rank(Q_{opt}).

Next, we show that b-rank$(p) \geq$ rank(Q_{opt}). Set $n :=$ b-rank(p). From the definition, there exist $2n$ polynomials $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n \in K[x]^{(k)}$ such that $p = \sum_{i=1}^n f_i g_i$. Suppose that $f_i(x) = \sum_{I \in \mathcal{I}_k} f_{i,I} x^I$, for $f_{i,I} \in K$, where $i = 1, 2, \ldots, n$. Then we can represent polynomials f_i by inner product of s_k-dimensional vectors as $f_i(x) = \boldsymbol{f}_i^\top \boldsymbol{v}(x)$, where $\boldsymbol{f}_i := (f_{i,I})_{I \in \mathcal{I}_k}$. Also, we can represent g_i as $g_i(x) = \boldsymbol{g}_i^\top \boldsymbol{v}(x)$ where $\boldsymbol{g}_i := (g_{i,I})_{I \in \mathcal{I}_k}$. Then we have

$$p(x) = \sum_{i=1}^n (\boldsymbol{f}_i^\top \boldsymbol{v}(x))(\boldsymbol{g}_i^\top \boldsymbol{v}(x)) = \sum_{i=1}^n \boldsymbol{v}(x)^\top (\boldsymbol{f}_i \boldsymbol{g}_i^\top) \boldsymbol{v}(x).$$

Defining $Q_0 := \sum_{i=1}^n \boldsymbol{f}_i \boldsymbol{g}_i^\top$, it follows that $\boldsymbol{v}(x)^\top Q_0 \boldsymbol{v}(x) = p(x)$. Thus Q_0 satisfies the constraints, and we have b-rank$(p) = n \geq$ rank$(Q_0) \geq$ rank(Q_{opt}). □

Observe that the feasible region of the above problem is an affine subspace of the set of matrices. We give similar formulations over symmetric and positive semidefinite matrices over $\mathbf{R}$.

Corollary 2.2. *For $p \in \mathbf{R}[x]^{(2k)}$,* b-rank$(p)$ *is at least the half of the optimum value of the following problem:*

$$\begin{array}{ll} \text{Minimize} & \operatorname{rank}(Q) \\ \text{subject to} & p(x) = \boldsymbol{v}(x)^\top Q \boldsymbol{v}(x), \\ & Q \in \operatorname{Sym}_{s_k}. \end{array}$$

Proof. Consider the optimum solution $Q' \in \operatorname{Mat}_n(\mathbf{R})$ of the corresponding optimization problem in Theorem 2.1. Then it holds that b-rank$(p) =$ rank(Q'). Since $Q = (Q' + Q'^\top)/2$ is a feasible solution of the above problem and rank$(Q') \geq$ rank$(Q)/2$, the statement holds. □

Corollary 2.3. *For $p \in \mathbf{R}[x]^{(2k)}$,* b-rank$(p)$ *is at least the half of the optimum value of the following problem:*

$$\begin{array}{ll} \text{Minimize} & \operatorname{rank}(Q_+) + \operatorname{rank}(Q_-) \\ \text{subject to} & p(x) = \boldsymbol{v}(x)^\top (Q_+ - Q_-) \boldsymbol{v}(x), \\ & Q_+, Q_- \in \operatorname{Psd}_{s_k}. \end{array}$$

Proof. Since any symmetric matrix $Q \in \mathrm{Sym}_n$ can be uniquely represented as the difference $Q = Q_+ - Q_-$ of the two positive semidefinite matrices $Q_+, Q_- \in \mathrm{Psd}_n$ satisfying $\mathrm{rank}(Q) = \mathrm{rank}(Q_+) + \mathrm{rank}(Q_-)$, the statement follows from Corollary 2.2. $\square$

For $p \in \mathbf{R}[x]^{(2k)}$, we define $\mathcal{X}_p$ as the set of pairs (Q_+, Q_-) of $s_k \times s_k$ matrices satisfying linear equation $p(x) = \boldsymbol{v}(x)^\top (Q_+ - Q_-)\boldsymbol{v}(x)$. By the embedding

$$(Q_+, Q_-) \mapsto \begin{pmatrix} Q_+ & O \\ O & Q_- \end{pmatrix}.$$

we regard $\mathcal{X}_p$ as an affine subspace of Sym_{2s_k}. Then $\mathcal{X}_p \cap \mathrm{Psd}_{2s_k}$ is the feasible region of the optimization problem in Corollary 2.3.

2.2 b-rank of generic polynomials

The inequality in Theorem 1.5 is nontrivial only if the bi-polynomial rank is larger than $2^{2k-1}(k-1)D^{k-1}$. We are going to show that for $D \gg k$ and a generic polynomial, this condition holds. Here we suppose that K is an algebraically closed field of characteristic zero. We use the terminologies in Section 2.1. Observe that polynomials in $K[x]^{(2k)}$ are determined by s_{2k} coefficients, and therefore we regard that a homogeneous polynomial $p \in K[x]^{(2k)}$ is a point in $K^{s_{2k}}$, under the correspondence $p(x) = \sum_{I \in \mathcal{I}_D^{(2k)}} a_I x^I \mapsto (a_I)_{I \in \mathcal{I}_D^{(2k)}} \in K^{s_{2k}}$. Then the set of polynomials p satisfying $\mathrm{b\text{-}rank}(p) \leq r$ are characterized in terms of algebraic geometry, as follows.

Theorem 2.4. *Let $S := \{q \in K[x]^{(2k)} \mid \mathrm{b\text{-}rank}(q) \leq r\} \subseteq K^{s_{2k}}$. Then the Zariski closure $\overline{S}$ is an irreducible variety having dimension at most $r(2s_k - r)$.*

Proof. We identify $\mathrm{Mat}_{s_k}(K)$ with $K^{s_k^2}$. Let $Z_r := \{X \in K^{s_k^2} \mid \mathrm{rank}(X) \leq r\}$. Z_r is called the *determinantal variety*, and it is known that Z_r is an irreducible variety of dimension $r(2s_k - r)$ (see, e.g., [4]). We define $\pi : K^{s_k^2} \to K^{s_{2k}}$ by

$$(\pi(X))_H := \sum_{I,J : I+J=H} X_{I,J} \quad (I, J \in \mathcal{I}_k,\ H \in \mathcal{I}_{2k}).$$

This π is a linear projection. For $q \in K[x]^{(2k)}$, Theorem 2.1 shows that $\mathrm{b\text{-}rank}(q) \leq r$ if and only if there exists $X \in K^{s_k^2}$ such that $\mathrm{rank}(X) \leq r$ and $\pi(X) = q$. Thus it follows that $S = \pi(Z_r)$. As the Zariski closure of the linear projection of the irreducible variety Z_r, $\overline{S} = \overline{\pi(Z_r)}$ is irreducible and its dimension is at most $r(2s_k - r)$. $\square$

From Theorem 2.4, we can obtain a lower bound of the bi-polynomial rank for generic polynomials.

Proposition 2.5. *For a polynomial $p \in \mathbf{C}[x]^{(2k)}$ with algebraically independent coefficients over $\mathbf{Q}$, it holds that $\mathrm{b\text{-}rank}(p) \geq k!D^k/2(2k)!$.*

Proof. Let $r := \mathrm{b\text{-}rank}(p)$ and $S_r := \{q \in \mathbf{C}[x]^{(2k)} \mid \mathrm{b\text{-}rank}(q) \leq r\} \subseteq \mathbf{C}^{s_{2k}}$. From Theorem 2.4, $\overline{S_r}$ is an irreducible variety of dimension at most $r(2s_k - r)$. Since Z_r and hence $\overline{S_r}$ is defined over $\mathbf{Q}$ and p has no algebraic relation over $\mathbf{Q}$ in its coefficients, $p \in \overline{S_r}$ implies that $\overline{S_r}$ must be $\mathbf{C}^{s_{2k}}$. Comparing the dimensions, it must holds that $r(2s_k - r) \geq s_{2k}$. This implies $r^2 - 2s_k r + s_{2k} \leq 0$ and

$$r \geq s_k - \sqrt{s_k^2 - s_{2k}} \geq s_k \left(1 - \left(1 - \frac{s_{2k}}{2s_k^2}\right)\right) = \frac{s_{2k}}{2s_k} \geq \frac{k!D^k}{2(2k)!}.$$

In the second inequality, we use the fact that $\sqrt{1-x} \leq 1 - x/2$ for $x \leq 1$. $\square$

This lower bound is asymptotically tight if k is a constant.

Proposition 2.6. *For any polynomial $p \in K[x]^{(2k)}$, it holds that* $\text{b-rank}(p) \le s_k \le D^k$.

This proposition immediately follows from Lemma 5.6.

2.3 b-rank and arithmetic branching program

We here discuss a relation between the bi-polynomial rank and an arithmetic branching program (ABP). We show that the following weaker statement than Theorem 1.5 easily follows from known facts on an ABP of determinant.

Proposition 2.7. *For a polynomial $p \in K[x]$, $k \in \mathbf{N}$, and $x_0 \in K^D \setminus \mathrm{Zeros}(p)$, it holds that*

$$\mathrm{dc}(p) \ge \sqrt{\text{b-rank}(p_{x_0}^{(2k)})}.$$

We omit the case $x_0 \in \mathrm{Zeros}(p)$ for the simplicity of the proof. We use the following lemma which is a variation of Lemma 5.1.

Lemma 2.8. *Let $p \in K[x]$ with $\mathrm{dc}(p) = n$. Then, for all $x_0 \in K^D \setminus \mathrm{Zeros}(p)$, there exist a linear polynomial matrix $A \in (K[x])^{n\times n}$ and $\alpha \in \mathbf{R}$ such that $p_{x_0}(x) = \alpha \det(A(x)+I)$.*

Proof. From the definition of the determinantal complexity, there exists an affine polynomial matrix $Q \in (K[x])^{n\times n}$ such that $p = \det(Q)$. In particular, given any $x_0 \in K^D \setminus \mathrm{Zeros}(p)$ it follows that $p_{x_0}(x) = \det(Q(x + x_0))$. Define $\alpha := \det(Q(x_0)) = p_{x_0}(0) \ne 0$. Since Q is an affine polynomial matrix, we can represent $Q(x + x_0) = L(x) + Q(x_0)$ where $L \in (K[x])^{n\times n}$ is a linear polynomial matrix. We have

$$\det(Q(x + x_0)) = \alpha \det(Q(x_0)^{-1})\det(L(x) + Q(x_0)) = \alpha\det(Q(x_0)^{-1}L(x) + I).$$

Since $Q(x_0)^{-1}L(x)$ is also a linear polynomial matrix, we define $A(x) := Q(x_0)^{-1}L(x)$, and then the statement follows. $\square$

We formally define an ABP discussed in Section 1.

Definition 2.9 (Nisan [12], see also [14])**.** An (homogeneous) arithmetic branching program (ABP) over $K[x]$ is a layered graph with $n + 1$ layers as follows. The layers are labeled by $0, 1, \ldots, n$. The edges of the graph go from layer i to layer $i + 1$. Every edge e is labeled by a (homogeneous) linear polynomial $\ell_e \in K[x]$. Layer 0 has only one vertex called the source, and layer n has only one vertex called the sink. For every directed path from the source to the sink $\gamma = (e_1, e_2, \ldots, e_n)$, define the polynomial f_γ associated to γ as $f_\gamma = \ell_{e_1}\ell_{e_2}\cdots\ell_{e_n}$. The polynomial computed by ABP is $\sum_\gamma f_\gamma$.

For an ABP $\mathcal{A}$, we define the width $w_k(\mathcal{A})$ of layer k as the number of vertices in the layer k. Given a homogeneous polynomial f with degree at least k, we denote by $w_k(f)$ the minimum $w_k(\mathcal{A})$ over ABPs $\mathcal{A}$ which compute f.

The following is an easy observation.

Fact 2.10. *For $f \in K[x]^{(2k)}$, it holds that* $\text{b-rank}(f) \le w_k(f)$.

Proof. Suppose that an ABP $\mathcal{A}$ computes f. Let V be the set of vertices in the layer k of $\mathcal{A}$. For $v \in V$, let $\mathcal{R}_v$ and $\mathcal{R}'_v$ be the sets of path from the source to v and from v to the sink, respectively. Then it holds that

$$f = \sum_{v\in V}\left(\sum_{\gamma\in\mathcal{R}_v} f_\gamma\right)\left(\sum_{\gamma'\in\mathcal{R}_v} f_{\gamma'}\right).$$

Therefore we have $\text{b-rank}(f) \le |V| = w_k(f)$. $\square$

The next statement is a well-known result.

Theorem 2.11 (Mahajan and Vinay [8]). *Let $A \in (K[x])^{n\times n}$ be a linear polynomial matrix, and $r \in [1, n-2]$. Then there exists an ABP $\mathcal{A}$ over $K[x]$ such that $\mathcal{A}$ computes the coefficient of λ^{n-r} in $\det(A(x)+\lambda I)$ and satisfying $w_k(\mathcal{A}) \le n^2$ for all $k \in [1, r-1]$.*

Then Proposition 2.7 is proved as follows.

Proof of Proposition 2.7. By Lemma 2.8, there exists a linear polynomial matrix $A \in (K[x])^{n\times n}$ and $\alpha \in \mathbf{R}$ such that $p_{x_0}(x) = \alpha \det(A(x)+I)$. Then $p_{x_0}^{(2k)}$ is equal to the coefficient of λ^{n-2k} in $\alpha \det(A(x)+\lambda I)$. Then by Theorem 2.11, there exists an ABP $\mathcal{A}$ with $w_k(\mathcal{A}) \le n^2$ which computes $p_{x_0}(x)^{(2k)}$. By Fact 2.10, we have $\text{b-rank}(p_{x_0}^{(2k)}) \le w_k(p_{x_0}^{(2k)}) \le w_k(\mathcal{A}) \le n^2 = \mathrm{dc}(p)^2$. ☐

3 Lower bounds of $\mathrm{dc}(\mathrm{perm}_d)$ by b-rank: case $k = 1$

Considering the case $k = 1$ in Theorem 1.5, we obtain lower bounds of $\mathrm{dc}(\mathrm{perm}_d)$ by the bi-polynomial rank. We define $\Sigma_d \in \mathrm{Zeros}(\mathrm{perm}_d)$ as follows:

$$\Sigma_d := \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 1 \\ 1 & \cdots & 1 & 1-d \end{pmatrix}.$$

This is the same matrix appearing in the proof of Theorem 1.3 in [10].

3.1 Mignon-Ressayre bound from b-rank

This section is devoted to the demonstration of the bi-polynomial rank as an extention of Hessian rank. We give an alternative proof of the result of Mignon and Ressayre (Theorem 1.3). By using the bi-polynomial rank, Theorem 1.3 immediately follows from Theorem 1.5 as follows.

An alternative proof of Theorem 1.3. For any $p \in K[x]$ and $x_0 \in \mathrm{Zeros}(p)$, we have

$$p_{x_0}^{(2)}(x) = \frac{1}{2} \sum_{1 \le i,j \le D} x_i x_j \left(\frac{\partial^2}{\partial x_i \partial x_j} p \right)\Big|_{x=x_0}.$$

We define the Hessian $H_{p,x_0} = (h_{i,j})$ of p at x_0 by $h_{i,j} := \left(\frac{\partial^2}{\partial x_i \partial x_j} p\right)\Big|_{x=x_0}$. By definition, $\text{b-rank}(p_{x_0}^{(2)})$ is equal to the minimum number n of bilinear forms $(\sum_{l=1}^D b_l^m x_l)(\sum_{l=1}^D c_l^m x_l)$ $(m = 1, 2, \ldots, n)$ whose sum is equal to $p_{x_0}^{(2)}$. We define the rank one matrices $A_m = (a_{i,j}^m)$ for $m = 1, 2, \ldots, n$, by $a_{i,j}^m := b_i^m c_j^m$. Let $A := \sum_{m=1}^n A_m$, and then it holds that $A + A^\top = H_{p,x_0}$. Therefore we have $\text{b-rank}(p_{x_0}^{(2)}) \ge \mathrm{rank}(A) \ge \frac{1}{2}\mathrm{rank}(H_{p,x_0})$. By Theorem 1.3, by putting $k = 1$ it holds that

$$\mathrm{dc}(p) \ge \text{b-rank}(p_{x_0}^{(2)}) \ge \frac{1}{2}\mathrm{rank}(H_{p,x_0}).$$

In the case of $p = \mathrm{perm}_d$, Mignon and Ressayre proved $\mathrm{rank}(H_{\mathrm{perm}_d,\Sigma_d}) = d^2$. Thus Theorem 1.3 follows from our Theorem 1.5. ☐

3.2 Lower bound of $\mathrm{dc}(\mathrm{perm}_d)$ over the field $\mathbf{R}$

Theorem 1.7 improves the current best lower bound given by Mignon and Ressayre. We present the proof in this section. Given a symmetric matrix, we denote by a tuple (n_+, n_-, n_0) the *signature* of the matrix, that is, the number of positive, negative, zero eigenvalues, respectively. If symmetric matrices S and S' have the same signature, we denote $S \sim S'$. By Sylvester's law of inertia, $S \sim S'$ if and only if $S \sim TST^\top$ for a nonsingular matrix T. We use the next lemma.

Lemma 3.1. *Let $Q \in \mathrm{Mat}_n(\mathbf{R})$, $Q_{\mathrm{sym}} := Q + Q^\top$, and (n_+, n_-, n_0) be the signature of Q_{sym}. Then it holds that* $\mathrm{rank}(Q) \geq \max\{n_+, n_-\}$.

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_{n_+}$ be the positive eigenvalues of Q_{sym}, and $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{n_+}$ be the corresponding eigenvectors which are orthogonal to each other. Let $V_+ \subseteq \mathbf{R}^n$ be the n_+-dimensional subspace spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{n_+}$. For any nonzero vector $\boldsymbol{u} = \sum_{i=1}^{n_+} a_i \boldsymbol{v}_i \in V_+$ where $a_1, a_2, \dots, a_{n_+} \in \mathbf{R}$, it holds that

$$2\boldsymbol{u}^\top Q \boldsymbol{u} = \boldsymbol{u}^\top Q_{\mathrm{sym}} \boldsymbol{u} = \sum_{i=1}^{n_+} a_i^2 \boldsymbol{v}_i^\top Q \boldsymbol{v}_i = \sum_{i=1}^{n_+} a_i^2 \lambda_i > 0.$$

The above inequality shows that $Q\boldsymbol{u} \neq 0$ for all nonzero vectors $\boldsymbol{u}$ in n_+-dimensional space V_+, and therefore $\mathrm{rank}(Q) \geq n_+$. The same argument is also true for eigenvectors with negative eigenvalues, and it holds that $\mathrm{rank}(Q) \geq n_-$. □

The proof of Theorem 1.7 is given as follows.

Proof of Theorem 1.7. From Theorem 1.5, for $k = 1$ we obtain that $\mathrm{dc}(\mathrm{perm}_d) \geq \text{b-rank}(\mathrm{perm}^{(2)}_{d,\Sigma_d})$ over $\mathbf{R}$. We consider that a matrix $A = (a_{(i,j),(i',j')})$ of size d^2 is indexed by pairs of integers $(i,j), (i',j')$ where $i, j, i', j' \in [1,d]$. Since

$$\mathrm{perm}^{(2)}_{d,\Sigma_d}(x) = \frac{1}{2} \sum_{i,j,i',j' \in [1,d]} x_{i,j} x_{i',j'} \left(\frac{\partial^2 \mathrm{perm}_d}{\partial x_{i,j} \partial x_{i',j'}} \right) \Bigg|_{x=\Sigma_d},$$

we define the Hessian matrix $H = (h_{(i,j),(i',j')})$ of perm_d at Σ_d by the following equation.

$$h_{(i,j),(i',j')} = h_{(i',j'),(i,j)} := \left(\frac{\partial^2 \mathrm{perm}_d}{\partial x_{i,j} \partial x_{i',j'}} \right) \Bigg|_{x=\Sigma_d}.$$

The corresponding optimization problem in Theorem 2.1 is equal to the following.

$$\begin{array}{ll} \text{Minimize} & \mathrm{rank}(Q) \\ \text{subject to} & Q + Q^\top = H, \\ & Q \in \mathrm{Mat}_{d^2}(\mathbf{R}). \end{array}$$

Let Q_{opt} be an optimum solution of the above problem. By Theorem 2.1 we have $\text{b-rank}(\mathrm{perm}^{(2)}_{d,\Sigma_d}) = \mathrm{rank}(Q_{\mathrm{opt}})$. Let (n_+, n_-, n_0) be the signature of $H \in \mathrm{Sym}_{d^2}$. Since $Q_{\mathrm{opt}} + Q_{\mathrm{opt}}^\top = H$ by Lemma 3.1 it follows that $\mathrm{rank}(Q_{\mathrm{opt}}) \geq n_-$. Therefore we obtain

$$\mathrm{dc}(\mathrm{perm}_d) \geq \text{b-rank}(\mathrm{perm}^{(2)}_{d,\Sigma_d}) = \mathrm{rank}(Q_{\mathrm{opt}}) \geq n_-.$$

We are going to prove $n_- = (d-1)^2 + 1$. As in [10], H can be calculated as

$$H = (d-3)! \begin{pmatrix} O & B & \cdots & B & C \\ B & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & O & B & C \\ B & \cdots & B & O & C \\ C & \cdots & C & C & O \end{pmatrix},$$

where B and C are the following matrices of size d:

$$B = \begin{pmatrix} 0 & -2 & \cdots & -2 & d-2 \\ -2 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & -2 & d-2 \\ -2 & \cdots & -2 & 0 & d-2 \\ d-2 & \cdots & d-2 & d-2 & 0 \end{pmatrix}, C = (d-2) \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Let S_d be the symmetric matrix of size d defined as follows.

$$S_d := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Then it holds that $B \sim -S_d$ and $C \sim S_d$. Let I_d be the identity matrix of size d. The signature of S_d is $(1, d-1, 0)$, since the rank of the matrix $(S_d + I_d)$ is one with the nonzero eigenvalue d. Define a nonsingular matrix T of size d^2 as follows.

$$T := \begin{pmatrix} I_d & O & \cdots & \cdots & O \\ O & I_d & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \cdots & I_d & O \\ -\frac{1}{d-2}CB^{-1} & -\frac{1}{d-2}CB^{-1} & \cdots & -\frac{1}{d-2}CB^{-1} & I_d \end{pmatrix}.$$

Then $H \sim THT^\top$, where

$$TQ_{\mathrm{sym}}T^\top = \begin{pmatrix} O & B & \cdots & O & O \\ B & O & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & B & O \\ B & \cdots & B & O & O \\ O & \cdots & \cdots & O & -\frac{d-1}{d-2}CB^{-1}C \end{pmatrix}.$$

Let H' be the upper-left principal submatrix of $THT^\top$ of size $d(d-1)$, which is represented as follows.

$$Q' = \begin{pmatrix} O & B & \cdots & B \\ B & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B & \cdots & C & O \end{pmatrix}.$$

Denote by (l_+, l_-, l_0) and (m_+, m_-, m_0) the signatures of $(-CB^{-1}C)$ and H', respectively. Then it holds that $n_- = l_- + m_-$. Since $-CB^{-1}C \sim -(BC^{-1})(CBC)(BC^{-1})^\top =$

$-B \sim S_d$, l_- is equal to $d-1$. On the other hand, it holds that $H' = S_{d-1} \otimes C$, where $\otimes$ denote the Kronecker product. Since the set of eigenvalues of $S_{d-1} \otimes B$ consists of the products of all pair of eigenvalues of S_{d-1} and B, the signature of $S_{d-1} \otimes B$ is $(2d-3, (d-2)(d-1)+1, 0)$. Therefore we have $n_- = (d-1) + ((d-2)(d-1)+1) = (d-1)^2 + 1$. □

4 Toward strong lower bounds via concave minimization

We formulate the bi-polynomial rank as the minimum matrix-rank over an affine subspace of matrices in Section 2.1. Unfortunately, few results are known for giving theoretical lower bounds for the rank minimization problem. In our case, the calculation of such a minimum rank is still difficult for $k \geq 2$. We propose an approach to bound the minimum rank below by using the framework of the concave minimization. In this section, we fix the field $K = \mathbf{R}$.

4.1 Concave minimization for bounding minimum rank below

The object of the concave minimization is to minimize a concave function over a convex set. This setting is studied in the area of global optimization [13]. We use this framework to obtain lower bounds of the minimum rank over a subset of positive semidefinite matrices.

As in Section 1, for $Y \in \mathrm{Sym}_n$ and $l \in [1, n]$, we denote by $\mu_l(Y)$ the sum of the smallest l eigenvalues of Y. For $\mathcal{X} \subseteq \mathrm{Sym}_n$, we define $\underline{\mathrm{rank}}(\mathcal{X}) := \min_{X \in \mathcal{X}} \mathrm{rank}(X)$. Then the next statement is immediate from the definition of positive semidefinite matrices.

Lemma 4.1. *Let $\mathcal{X} \subseteq \mathrm{Psd}_n$, and $r \in \mathbf{N}$. Then $\underline{\mathrm{rank}}(\mathcal{X}) > r$ if and only if $\mu_{n-r}(X) > 0$ for all $X \in \mathcal{X}$.*

This statement suggests the way to solve the rank minimization problem over positive semidefinite matrices by minimizing μ_l over the feasible region. The computation of μ_l is formulated as the optimum solution of a semidefinite programming.

Proposition 4.2 (See [1, Section 4.1])**.** *Let $A \in \mathrm{Sym}_n$ and $l \in [1, n]$. Then $\mu_l(A)$ is equal to the optimum value of the following problem:*

$$\begin{array}{ll} \text{Minimize} & \mathrm{tr}(AX) \\ \text{subject to} & \mathrm{tr}(X) = l, \\ & X, I - X \in \mathrm{Psd}_n. \end{array}$$

Proof. Since A is a symmetric matrix, A is diagonalizable by some orthogonal matrix V, as $VAV^\top =: \tilde{A}$. Let $\tilde{X} := VXV^\top$, and then the replacement of A, X by $\tilde{A}, \tilde{X}$ does not change the optimum value. Therefore, without loss of generality, we can assume that A is a diagonal matrix with diagonal entries $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then the optimum value is attained by such X that the first l diagonal entries are 1, and the other entries are 0. It holds that the optimum value is $\sum_{i=1}^{l} \lambda_i = \mu_l(A)$. □

For latter use, we prepare the next statement.

Corollary 4.3. *Let $Y \in \mathrm{Sym}_n$. Then $\mu_l(Y)$ is at least $lz - \mathrm{tr}(Z)$ for $Z \in \mathrm{Psd}_n$ and $z \in \mathbf{R}$ satisfying $(Y + Z - zI) \in \mathrm{Psd}_n$.*

Proof. Observe that the following optimization problem is the dual of the problem in Proposition 4.2.

$$\begin{array}{ll} \text{Maximize} & lz - \operatorname{tr}(Z) \\ \text{subject to} & z \in \mathbf{R}, \\ & Z, Y + Z - zI \in \mathrm{Psd}_n. \end{array}$$

By the weak duality of semidefinite programming, for any feasible solution $(Z, z) \in \mathrm{Psd}_n \times \mathbf{R}$, the objective value $lz - \operatorname{tr}(Z)$ is at most $\mu_l(Y)$. □

The concavity of μ_l is proved as follows.

Corollary 4.4 (See [1, Section 4.1])**.** *For $l \in [1, n]$, $\mu_l : \mathrm{Sym}_n \to \mathbf{R}$ is a concave function.*

Proof. Let $X, Y \in \mathrm{Sym}_n$. Denote by $\mathrm{prb}(Y)$ the optimization problem in Proposition 4.2 for Y. Then an optimum solution of $\mathrm{prb}(\frac{X+Y}{2})$ is a feasible solution of both $\mathrm{prb}(X)$ and $\mathrm{prb}(Y)$, and therefore the optimum value of $\mathrm{prb}(\frac{X+Y}{2})$ is at least the average of optimum values of $\mathrm{prb}(X)$ and $\mathrm{prb}(Y)$. By Proposition 4.2, this indicates that $\frac{1}{2}(\mu_l(X)+\mu_l(Y)) \le \mu_l(\frac{X+Y}{2})$. □

In the theory of the concave minimization, the outer approximation approach [5, 15] (see also [13]) obtain a lower bound of the minimum of a given concave function by approximating the feasible region from outside. Given a set $\mathcal{Y} \subseteq \mathrm{Sym}_n$, we denote by $\operatorname{conv}(\mathcal{Y})$ the convex hull of $\mathcal{Y}$. In general, given a concave function f over $\mathcal{Y}$, it holds that $\min_{Y\in\mathcal{Y}} f(Y) = \min_{Y\in\operatorname{conv}(\mathcal{Y})} f(Y)$. Therefore the next statement holds.

Theorem 4.5. *Let $\mathcal{X} \subseteq \mathrm{Psd}_n$ and $r \in \mathbf{N}$. If there exists $\mathcal{Y} \subseteq \mathrm{Sym}_n$ satisfying the following property, then $\underline{\operatorname{rank}}(\mathcal{X}) > r$.*

(i) $\mathcal{X} \subseteq \operatorname{conv}(\mathcal{Y})$.

(ii) $\mu_{n-r}(Y) > 0$ for all $Y \in \mathcal{Y}$.

Proof. Since μ_{n-r} is a concave function, $0 < \min_{Y\in\mathcal{Y}} \mu_{n-r}(Y) = \min_{Y\in\operatorname{conv}(\mathcal{Y})} \mu_{n-r}(Y) \le \min_{X\in\mathcal{X}} \mu_{n-r}(X)$. By Lemma 4.1, $\mu_{n-r}(X) > 0$ for all $X \in \mathcal{X}$ implies $\underline{\operatorname{rank}}(\mathcal{X}) > r$. □

To utilize Theorem 4.5 for lower bounds of the bi-polynomial rank, we prove Proposition 1.8.

Proof of Proposition 1.8. By Corollary 2.3, $\text{b-rank}(p) \ge \frac{1}{2}\underline{\operatorname{rank}}(\mathcal{X}_p \cap \mathrm{Psd}_n)$. Then by Theorem 4.5, $\underline{\operatorname{rank}}(\mathcal{X}_p \cap \mathrm{Psd}_n) > r$, and therefore the statement holds. □

Corollary 4.3 may help the verification $\mu_{n-r}(Y) > 0$ as follows.

Corollary 4.6. *Let $p \in K[x]^{(2k)}$ and $r \in \mathbf{N}$. If there exists $\mathcal{Y} \subseteq \mathcal{X}_p$ satisfying the following property, then $\text{b-rank}(p) > r/2$.*

(i) $\mathcal{X}_p \cap \mathrm{Psd}_{2s_k} \subseteq \operatorname{conv}(\mathcal{Y})$.

(ii) For all $(Y_1, Y_2) \in \mathcal{Y}$, there exists $(Z_1, Z_2, z) \in \mathrm{Psd}_{s_k} \times \mathrm{Psd}_{s_k} \times \mathbf{R}$ such that $(Y_1 + Z_1 - zI), (Y_2 + Z_2 - zI) \in \mathrm{Psd}_{s_k}$ and $(2s_k - r)z - \operatorname{tr}(Z_1 + Z_2) > 0$.

Proof. The statement directly follows from Proposition 1.8 and Corollary 4.3. □

4.2 An explicit representation of $\mathcal{X}_{\mathrm{perm}_{d,\Sigma_d}^{(2k)}}$

The previous section discusses a general framework for giving lower bounds of the bi-polynomial rank. In the framework, to calculate b-rank(p) of $p \in K[x]^{(2k)}$, we consider the minimum of the concave function μ_{2s_k-r} over $\mathcal{X}_p \cap \mathrm{Psd}_{2s_k}$. Our final target is to obtain lower bounds of $\mathrm{dc}(\mathrm{perm}_d)$. In this section, we fix $p = \mathrm{perm}_{d,\Sigma_d}^{(2k)}$, and give an explicit representation of a projection $\mathcal{Z}_{2k}$ of $\mathcal{X}_p$. p is a multilinear polynomial, and to extract this feature, we define the subset $\mathcal{J}_{k,d^2} := \mathcal{I}_{k,d^2} \cap \{0,1\}^{d^2}$ of d^2-tuples. Define $d' := d-1$. Observe that Σ_d has good symmetry except for dth row/column. For $I = (i_{1,1}, i_{1,2}, \ldots, i_{d',d'}) \in \mathcal{J}_{k,d'^2}$, we define $\iota(I) \in \mathcal{J}_{k,d^2}$ by the insertion of zeros into the entries not in $\mathcal{J}_{k,d'^2}$. More concretely,

$$\iota(i_{1,1}, i_{1,2}, \ldots, i_{d',d'}) := (i_{1,1}, \ldots, i_{1,d-1}, 0, i_{2,1}, \ldots, i_{d',d'}, 0, \ldots, 0).$$

Let t_k be the cardinality of $\mathcal{J}_{k,d'^2}$. Then we define the projection $\pi : \mathrm{Sym}_{s_k} \to \mathrm{Sym}_{t_k}$ by $\pi(Y)_{I,J} := \alpha Y_{\iota(I),\iota(J)}$ for $I, J \in \mathcal{J}_{k,d'^2}$, where $\alpha := -\frac{1}{2k(d-2k-1)!}$ is a constant. We define $\mathcal{Z}_{2k} \subseteq \mathrm{Sym}_{t_k} \times \mathrm{Sym}_{t_k}$ as a projection of $\mathcal{X}_p$ as follows:

$$\mathcal{Z}_{2k} := \{(\pi(X_+), \pi(X_-)) \mid (X_+, X_-) \in \mathcal{X}_p\}.$$

Then it holds that

$$\begin{aligned}\underline{\mathrm{rank}}(\mathcal{Z}_{2k} \cap \mathrm{Psd}_{2t_k}) &\leq \underline{\mathrm{rank}}(\mathcal{X}_p \cap \mathrm{Psd}_{2s_k}) \\ &\leq \underline{\mathrm{rank}}(\mathcal{Z}_{2k} \cap \mathrm{Psd}_{2t_k}) + (s_k - t_k),\end{aligned}$$

where $s_k - t_k = O(d^{2k-2})$. Hence $\underline{\mathrm{rank}}(\mathcal{X}_p \cap \mathrm{Psd}_{2s_k}) = \Omega(d^{2k})$ if and only if $\underline{\mathrm{rank}}(\mathcal{Z}_{2k} \cap \mathrm{Psd}_{2t_k}) = \Omega(d^{2k})$. We are going to give an explicit representation of $\mathcal{Z}_{2k}$. In p, no monomial with repetition of a row/column index appear. To express this, we classify $\mathcal{J}_{2k,d'^2}$ into $\mathcal{H}_1$ and $\mathcal{H}_0$ as follows. We define $\mathcal{H}_1$ as the set of tuples $(H_{1,1}, H_{1,2}, \ldots, H_{d',d'}) \in \mathcal{J}_{2k,d'^2}$ satisfying $\sum_{j'=1}^{d'} H_{i,j'} \leq 1$ and $\sum_{i'=1}^{d'} H_{i',j} \leq 1$ for $i, j \in [1, d']$, and $\mathcal{H}_0 := \mathcal{J}_{2k,d'^2} \setminus \mathcal{H}_1$. Then $\mathcal{Z}_{2k}$ is given as the set of pairs (U, V) of matrices in Sym_{t_k} satisfying linear equations for all $H \in \mathcal{J}_{2k,d'^2}$:

$$\sum_{\substack{I,J \in \mathcal{J}_{k,d'^2} \\ I+J=H}} (u_{I,J} - v_{I,J}) = \begin{cases} 1, & H \in \mathcal{H}_1 \\ 0, & H \in \mathcal{H}_0 \end{cases}$$

Observe that this affine space $\mathcal{Z}_{2k}$ is represented by simple linear equations with coefficients in $\{0, \pm 1\}$.

5 Proof of Theorem 1.5

A *linear polynomial matrix* over $x_1, x_2, \ldots, x_D$ of size n is an $n \times n$ matrix $A(x) = (a_{i,j}(x)) \in (K[x])^{n \times n}$, where each element $a_{i,j}(x)$ is a (homogeneous) linear polynomial for $1 \leq i, j \leq n$. Denote by Λ_n^r the diagonal matrix of size n with diagonal entries $(0, \ldots, 0, \underbrace{1, \ldots, 1}_{r})$. For $\alpha, \beta \in \mathbf{Z}$ with $\alpha \leq \beta$, we denote $\{\alpha, \alpha+1, \ldots, \beta\}$ by $[\alpha, \beta]$.

Lemma 5.1. *Let $p \in K[x]$ with $\mathrm{dc}(p) = n$. Then, for all $x_0 \in \mathrm{Zeros}(p)$, there exist a linear polynomial matrix $A \in (K[x])^{n \times n}$ and $r \in [0, n-1]$ such that $p_{x_0}(x) = \det(A(x) + \Lambda_n^r)$.*

Proof. From the definition of the determinantal complexity, there exists an affine polynomial matrix $Q \in (K[x])^{n\times n}$ such that $p = \det(Q)$. In particular, given any $x_0 \in \mathrm{Zeros}(p)$ it follows that $p_{x_0}(x) = \det(Q(x + x_0))$. Observe that $\det(Q(x_0)) = p_{x_0}(0) = 0$ and $\mathrm{rank}(Q(x_0)) \le n-1$. Let $r \in [0, n-1]$ be the rank of $Q(x_0)$. Then there exist nonsingular matrices S, T of size n such that $SQ(x_0)T = \Lambda_n^r$ and $\det(ST) = 1$. Since Q is an affine polynomial matrix, we can represent $Q(x + x_0) = L(x) + Q(x_0)$ where $L \in (K[x])^{n\times n}$ is a linear polynomial matrix. We have

$$\det(Q(x + x_0)) = \det(L(x) + Q(x_0)) = \det(S(L(x) + Q(x_0))T) = \det(SL(x)T + \Lambda_n^r).$$

Since $SL(x)T$ is also a linear polynomial matrix, we define $A(x) := SL(x)T$, and then the statement follows. □

Given a linear polynomial matrix $A \in (K[x])^{n\times n}$, $k \in \mathbf{N}$ and $r \in [0, n-1]$, we define

$$p_{A,k,r}(x) := (\det(A(x) + \Lambda_n^r))^{(k)}.$$

Then it holds that $\det(A(x)+\Lambda_n^r) = \sum_{k=n-r}^{n} p_{A,k,r}(x)$. The next statement is the essence of our result.

Proposition 5.2. *For $A \in (K[x])^{n\times n}$, it holds that* $\text{b-rank}(p_{A,2k,n-1}) \le n+2(k-1)D^{k-1}$.

The proof of Proposition 5.2 is given in the next section. By this proposition, the following statement holds.

Lemma 5.3. *(i) For $r \in [n-2k, n-1]$, it holds that* $\text{b-rank}(p_{A,2k,r}) \le 2^{n-r-1}(n+2(k-1)D^{k-1})$.

(ii) For $r = n-2k$, it holds that $\text{b-rank}(p_{A,2k,r}) \le \binom{2k}{k}$.

Proof. (i), let $i := n - r$. We prove the statement by the induction on i. The case $i = 1$ directly follows from Proposition 5.2. Suppose that for $i \le 2k-1$, the statement holds. Denote by $A' \in K[x]^{n-1\times n-1}$ the linear polynomial matrix obtained by deleting the $(i+1)$th row and column of A. Since determinant is a bi-linear form, we have

$$\det(A(x) + \Lambda_n^{n-i}) = \det(A(x) + \Lambda_n^{n-(i+1)}) + \det(A'(x) + \Lambda_{n-1}^{(n-1)-i}),$$

and therefore $p_{A,2k,n-(i+1)} = p_{A,2k,n-i} - p_{A',2k,(n-1)-i}$. By inductive hypothesis, we have

$$\begin{aligned}&\text{b-rank}(p_{A,2k,n-(i+1)}) \le \text{b-rank}(p_{A,2k,n-i}) + \text{b-rank}(p_{A',2k,(n-1)-i})\\ &\le 2^{i-1}(n + 2(k-1)D^{k-1}) + 2^{i-1}((n-1) + 2(k-1)D^{k-1}) \le 2^i(n + 2(k-1)D^{k-1}).\end{aligned}$$

Therefore the statement (i) holds.

(ii) We have

$$p_{A,2k,n-2k}(x) = (\det(A(x) + \Lambda_n^{n-2k}))^{(2k)} = \det(A_{2k}(x)),$$

where A_{2k} is the leading principal submatrix of A of size $2k$. Given $I \subseteq [1, 2k]$ with $|I| = k$, we denote by B_I the square submatrix of A_{2k} consisting of rows and columns corresponding to indices I and $[1, k]$, respectively. Also, we denote by $B_{\bar{I}}(x)$ the square submatrix of $A_{2k}(x)$ with row and column indices $[1, 2k] \setminus I$ and $[k+1, 2k]$, respectively. Then the following is an elementary formula for determinant.

$$\det(A_{2k}) = \sum_{I\subseteq[1,2k]:|I|=k} \mathrm{sign}(I)\det(B_I)\cdot\det(B_{\bar{I}}),$$

where $\mathrm{sign}(I) \in \{1, -1\}$. Since $\det(B_I), \det(B_{\bar{I}}) \in K[x]^{(k)}$ for all $I \subseteq [1, 2k]$ with $|I| = k$, we have $\text{b-rank}(p_{A,2k,r}) \le \binom{2k}{k}$. □

We give a proof of Theorem 1.5.

Proof of Theorem 1.5. Suppose that $\mathrm{dc}(p) = n$. By Lemma 5.1, for all $x_0 \in \mathrm{Zeros}(p)$, there exist a linear polynomial matrix $A \in (K[x])^{n\times n}$ and $r \in [0, n-1]$ such that $p_{x_0}(x) = \det(A(x) + \Lambda_n^r)$. Then $p_{x_0}^{(2k)}(x) = (\det(A(x) + \Lambda_n^r))^{(2k)} = p_{A,2k,r}$. If $r < n - 2k$ or $n = 1$, then $p_{A,2k,r} = 0$ and $\text{b-rank}(p_{x_0}^{(2k)}) = 0$. The statement is trivial in this case, and we assume that $r \in [n-2k, n-1]$ and $n \geq 2$. By Lemma 5.3 it holds that

$$\text{b-rank}(p_{A,2k,r}) \leq \max\{2^{2k-2}(n + 2(k-1)D^{k-1}), \binom{2k}{k}\} \leq 2^{2k-2}(n + 2(k-1)D^{k-1}),$$

since $k \leq D$, $n \geq 2$, and $\binom{2k}{k} \leq (2k)^k \leq 2^{2k-2}(n + 2(k-1)D^{k-1})$. Therefore it holds that

$$\frac{1}{2^{2k-2}}\,\text{b-rank}(p_{x_0}^{(2k)}) - 2(k-1)D^{k-1} \leq n = \mathrm{dc}(p).$$

□

Proof of Proposition 5.2

We denote $p_{A,k,n-1} := (\det(A(x) + \Lambda_n^{n-1}))^{(k)}$ by $p_{A,k}$ for notational simplicity. Let us define the following notions about clows and cycles on a vertex set. The former is a terminology of Mahajan and Vinay [8].

- Let $V_n := [1, n]$, and we call elements of V_n as *vertices*.
- A *clow* (standing for closed walk) on V_n is an ordered tuple of vertices $\langle v_1, v_2, \ldots, v_l\rangle$ such that $v_1 < v_i$ for $i = 2, \ldots, l$. The vertex v_1 is referred to as the *head* of c, and l is called the *length* of c. If all vertices $v_1, v_2, \ldots, v_l$ are distinct, the clow is particularly called a *cycle*. Note that $\langle v\rangle$ is a cycle for all $v \in V_n$.
- Given a clow $c = \langle v_1, v_2, \ldots, v_l\rangle$ and a linear polynomial matrix $A(x) := (a_{i,j}(x)) \in (K[x])^{n\times n}$, $a_c \in K[x]^{(l)}$ is defined as $a_c := \prod_{i=1}^{l} a_{v_i,v_{i+1}}$, with identification $v_{l+1} = v_1$.
- A *clow sequence* $C = (c_1, c_2, \ldots, c_m)$ is an ordered tuple of clows $c_1, \ldots, c_m$, where the head of c_i is strictly less than the head of c_j if $i < j$. The *size* $\ell(C)$ of C is defined by $\ell(C) := m$.
- We denote by $\overline{\mathcal{C}}_{n,k}$ the set of clow sequences C on V_n such that the sum of length of all clows in C is k. The subset $\mathcal{C}_{n,k} \subseteq \overline{\mathcal{C}}_{n,k}$ consists of vertex disjoint clow sequences C' where each clow in C' is a cycle. Since every element C' of $\mathcal{C}_{n,k}$ includes exactly k distinct vertices, we call C' as a *cycle k-cover*.
- The *sign* of $C \in \overline{\mathcal{C}}_{n,k}$ is defined as $(-1)^{n+\ell(C)}$.
- We denote by $\overline{\mathcal{C}}_{n,k,1} \subseteq \overline{\mathcal{C}}_{n,k}$ the set of clow sequences which includes the vertex $1 \in V_n$. Also, $\mathcal{C}_{n,k,1} \subseteq \overline{\mathcal{C}}_{n,k,1}$ is defined as the set of cycle k-covers which include the vertices $1 \in V_n$. Given $C \in \overline{\mathcal{C}}_{n,k,1}$, we denote by c_1 the unique clow in C which includes the vertex 1.
- Given a linear polynomial matrix $A(x) = (a_{i,j}(x)) \in (K[x])^{n\times n}$ and a clow sequence $C = (c_1, c_2, \ldots, c_{\ell(C)}) \in \overline{\mathcal{C}}_{n,k}$, we define a polynomial $a_C \in K[x]^{(k)}$ by $a_C := \prod_{i=1}^{\ell(C)} a_{c_i}$.

Lemma 5.4. *Given a linear polynomial matrix $A \in (K[x])^{n\times n}$, it holds that*

$$p_{A,k}(x) = (-1)^{n-k} \sum_{C\in\mathcal{C}_{n,k}} \mathrm{sign}(C)a_C$$

Proof. We expand $\det(A(x) + \Lambda_n^{n-1}) = \sum_{i=1}^n p_{A,i}(x)$ with respect to $a_{i,j}(x)$. Given U with $\{1\} \subseteq U \subseteq V_n$, A_U is defined as the principal submatrix of A consisting of rows and columns having indices in U. Since determinant is a multilinear function and Λ_n^{n-1} has nonzero entries which is equal to 1 only on diagonal, it follows that

$$\det(A(x) + \Lambda_n^r) = \sum_{U:\{1\}\subseteq U\subseteq V_n} \det(A_U(x)) = \sum_{k=1}^{n} \sum_{\substack{U:|U|=k\\ \{1\}\subseteq U\subseteq V_n}} \det(A_U(x)). \tag{5.1}$$

For $U \subseteq V_n$, denote by $\mathcal{C}_{n,U} \subseteq \mathcal{C}_{n,|U|}$ the set of cycle $|U|$-covers which include all vertices in U. Since elements in $\mathcal{C}_{n,U}$ consist of vertices in U, we can regard them as cycle $|U|$-covers on U. Let $\mathfrak{S}_U$ be the set of permutations on the finite set U. Observe that there is a natural one-to-one correspondence between permutations $\sigma \in \mathfrak{S}_U$ and cycle $|U|$-covers $C \in \mathcal{C}_{n,U}$, satisfying $\mathrm{sign}(\sigma) = (-1)^{n-|U|}\,\mathrm{sign}(C)$ and $\prod_{i\in U} a_{i,\sigma(i)} = a_C$. Therefore from the definition of determinant it follows that

$$\det(A_U) = \sum_{\sigma\in\mathfrak{S}_U} \mathrm{sign}(\sigma) \prod_{i\in U} a_{i,\sigma(i)} = (-1)^{n-|U|} \sum_{C\in\mathcal{C}_{n,U}} \mathrm{sign}(C)a_C. \tag{5.2}$$

By (5.1), (5.2), we obtain

$$\begin{aligned}\det(A(x) + \Lambda_n^r) &= \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{U:|U|=k\\ \{1\}\subseteq U\subseteq V_n}} \sum_{C\in\mathcal{C}_{n,U}} \mathrm{sign}(C)a_C(x)\\ &= \sum_{k=1}^{n} (-1)^{n-k} \sum_{C\in\mathcal{C}_{n,k,n-r}} \mathrm{sign}(C)a_C(x) = \sum_{k=1}^{n} (-1)^{n-k} p_{A,k,r}(x).\end{aligned}$$

The second equation holds since $\mathcal{C}_{n,k,n-r}$ is the disjoint union of $\mathcal{C}_{n,U}$ over U satisfying $\{1\} \subseteq U \subseteq V_n$ and $|U| = k$. Therefore it holds that $p_{A,k} = (\det(A(x) + \Lambda_n^r))^{(k)} = (-1)^{n-k}p_{A,k,r}(x)$. ∎

The following lemma was essentially given in [18, Section 3], and proved in full detail in [8, 9].

Lemma 5.5. *It holds that*

$$p_{A,k} = (-1)^{n-k} \sum_{C\in\overline{\mathcal{C}}_{n,k,1}} \mathrm{sign}(C)a_C.$$

Proof. Since $\mathcal{C}_{n,k,1} \subseteq \overline{\mathcal{C}}_{n,k,1}$, by the definition of $p_{A,k,1}$ it is enough to show that

$$\sum_{C\in\overline{\mathcal{C}}_{n,k,1}\setminus\mathcal{C}_{n,k,1}} \mathrm{sign}(C)a_C(x) = 0. \tag{5.3}$$

Observe that $C \in \overline{\mathcal{C}}_{n,k,1}$ is not in $\mathcal{C}_{n,k,1}$ if and only if C has a repetition of vertices. Suppose that $C = (c_1, c_2, \ldots, c_m) \in \overline{\mathcal{C}}_{n,k,1}$ have a repetition of vertices. Let $l \le m$ be the maximum number such that $(c_l, \ldots, c_m)$ has a repetition but $(c_{l+1}, \ldots, c_m)$ does not.

Let $c_l = \langle v_1, v_2, \ldots, v_i\rangle$ and v_j be the first element in c_l which is either (1) equal to one of v_t where $1 < t < j$, or (2) equal to an element in one of $c_{l+1}, \ldots, c_m$. Precisely one of them will occur.

In the case (1), let $\langle u_t, u_{t+1}, \ldots, u_{j-1}\rangle$ be the clow obtained by cyclically reordering the vertex sequence $v_t, v_{t+1} \ldots, v_{j-1}$ (these vertices are all distinct and there is a unique minimum element). We replace c_l by two clows $\langle v_1, \ldots, v_{t-1}, v_j, \ldots, v_i\rangle$ and $\langle u_t, \ldots, u_{j-1}\rangle$, which are made by separation around v_t.

In the case (2), let $c' = \langle u_1, \ldots, u_{s-1}, v_j, u_{s+1}, \ldots, u_{i'}\rangle$ be the clow including v_j. We replace c_l and c' by the single clow $\langle v_1, \ldots, v_j, u_{s+1}, \ldots, u_{i'}, u_1, \ldots, u_{s-1}, v_j, v_{j+1}, \ldots, v_i\rangle$, which is constructed by the insertion of c' into c around the common vertex v_t. It can be verified that the procedures (1) and (2) are inverses of each other, which result in a one-to-one correspondence in clow sequences with repetition. If $C \in \overline{\mathcal{C}}_{n,k,1} \backslash \mathcal{C}_{n,k,1}$ is converted to $C' \in \overline{\mathcal{C}}_{n,k,1} \backslash \mathcal{C}_{n,k,1}$ by the above procedure, then $a_C(x) = a_{C'}(x)$ and $\operatorname{sign}(C) = -\operatorname{sign}(C')$. Thus the equation (5.3) holds. $\square$

The number of polynomials needed to span $K[x]^{(k)}$ as a K-vector space is $\binom{D+k-1}{k}$, and we define $s_k := \binom{D+k-1}{k}$. The following is a general property of polynomials.

Lemma 5.6. *For $p \in K[x]^{(k)}$ and $m \le k$, there exist $2s_m$ polynomials $f_1, f_2, \ldots, f_{s_m} \in K[x]^{(m)}$ and $g_1, g_2, \ldots, g_{s_m} \in K[x]^{(k-m)}$ such that $p = \sum_{i=1}^{s_m} f_i g_i$.*

Proof. Let $P_m := \{x_1^{j_1} x_2^{j_2} \cdots x_D^{j_D} \mid j_1, j_2, \ldots, j_D \in \mathbf{Z}_+,\ \sum_{l=1}^{D} j_l = m\}$ be the set of monomials with degree m. Take distinct $f_i \in P_m$ for $i = 1, 2, \ldots, s_m$. Since each term in p is divisible by at least one of f_i, we can obtain a decomposition $p = \sum_{i=1}^{s_m} f_i g_i$. $\square$

For $t = 1, 2, \ldots, 2k$, let $q_{A,2k,t} \in K[x]^{(2k)}$ be defined by

$$q_{A,2k,t} := \sum_{\substack{C \in \overline{\mathcal{C}}_{n,2k,1} \\ |c_1| = t}} \operatorname{sign}(C) a_C,$$

where c_1 is the unique clow in C including the vertex 1, and $|c_1|$ is the length of c_1. Then it holds that $p_{A,2k} = (-1)^{n-2k} \sum_{t=1}^{2k} q_{A,2k,t}$. Let $\mathcal{F}_t$ be the set of clows with length t which include the vertex $1 \in V_n$.

Lemma 5.7. *(i) For $t = 2k$, it holds that* $\operatorname{b-rank}(q_{A,2k,2k}) \le n - 1$.

(ii) For $t = 1, 2, \ldots, 2k-1$, let $t' := \min\{t, 2k-t\}$. Then it holds that $\operatorname{b-rank}(q_{A,2k,t}) \le s_{k-t'}$.

Proof. (i) We have $q_{A,2k,2k} = (-1)^{n+1} \sum_{c \in \mathcal{F}_{2k}} a_c$. For $v = 2, 3, \ldots, n$, let $\mathcal{F}_{2k,v} \subseteq \mathcal{F}_{2k}$ be the set of clows whose $(k+1)$th vertices are equal to v. Note that the $(k+1)$th vertex of a clow $c \in \mathcal{F}$ is one of $\{2, 3, \ldots, n\}$, since the head of c is 1. Then we have $q_{A,2k,2k} = (-1)^{n+1} \sum_{v \in [2,n]} \sum_{c \in \mathcal{F}_{2k,v}} a_c$. Let $\mathcal{R}_v$ and $\mathcal{R}'_v$ be the sets of $k+1$ vertex sequences $R = (1, u_2, u_3, \ldots, u_k, v)$ and $R' = (v, u'_2, u'_3 \ldots, u'_k, 1)$, respectively, such that $u_2, u_3, \ldots, u_m,\ u'_2, u'_3, \ldots, u'_{k+1} \in \{2, 3, \ldots, n\}$. We define $a_R := \prod_{i=1}^{k} a_{u_i, u_{i+1}}$ for $R = (u_1, u_2, \ldots, u_{k+1})$ in $\mathcal{R}_v$ or $\mathcal{R}'_v$. Then there is a one-to-one correspondence between $\mathcal{F}_{2k,v}$ and $\mathcal{R}_v \times \mathcal{R}'_v$ by the following correspondence

$$\begin{aligned} &\mathcal{F}_{2k,v} \ni \langle 1, u_2, u_3, \ldots, u_k, v, u_{k+2}, u_{k+3}, \ldots, u_{2k}\rangle \\ &\mapsto ((1, u_2, u_3, \ldots, u_k, v), (v, u_{k+2}, u_{k+3}, \ldots, u_{2k}, 1)) \in \mathcal{R}_v \times \mathcal{R}'_v. \end{aligned}$$

Therefore it holds that

$$\sum_{c \in \mathcal{F}_{k,v}} a_c = \left(\sum_{R \in \mathcal{R}_v} a_R \right) \left(\sum_{R' \in \mathcal{R}'_v} a_{R'} \right).$$

Therefore it holds that b-rank$(q_{A,2k,2k}) \leq \sum_{v \in [2,n]}$ b-rank$(\sum_{c \in \mathcal{F}_{k,v}} a_c) \leq n-1$.

(ii) Let $\mathcal{G}_{2k-t} \subseteq \overline{\mathcal{C}}_{n,2k-t}$ be the set of clow sequences which do not include the vertex $1 \in V_n$. Observe that a clow sequence $C \in \overline{\mathcal{C}}_{n,2k,1}$ with $|c_1| = t$ is uniquely determined by a pair of a clow $c_1 \in \mathcal{F}_t$ and a clow sequence $C' \in \mathcal{G}_{k-t}$. Furthermore, both can be chosen independently. Therefore we have

$$q_{A,2k,t} = \left(\sum_{c \in \mathcal{F}_t} a_c \right) \left(- \sum_{C' \in \mathcal{G}_{2k-t}} \operatorname{sign}(C') a_{C'} \right).$$

One of polynomial $(\sum_{c \in \mathcal{F}_t} a_c)$ and $(-\sum_{C' \in \mathcal{G}_{2k-t}} \operatorname{sign}(C') a_{C'})$ has degree t', and we denote it by α_1. The other is denoted by α_2. If $t' = k$, we already have the decomposition $q_{A,2k,t} = \alpha_1 \alpha_2$, and b-rank$(q_{A,2k,t}) = 1 \leq s_{k-t'}$. Otherwise, by Lemma 5.6 there exist $s_{k-t'}$ polynomials $\beta_1, \beta_2, \ldots, \beta_{s_{k-t'}} \in K[x]^{(k-t')}$ and $\gamma_1, \gamma_2, \ldots, \gamma_{s_{k-t'}} \in K[x]^{(k)}$ such that $\alpha_2 = \sum_{i=1}^{s_{k-t'}} \beta_i \gamma_i$. Then we have

$$q_{A,2k,t} = \sum_{i=1}^{s_{k-t'}} (\alpha_1 \beta_i) \cdot \gamma_i,$$

and b-rank$(q_{A,2k,t}) \leq s_{k-t'}$. □

Proof of Proposition 5.2. By definition, $p_{A,2k} = (-1)^{n-2k} \sum_{t=1}^{2k} q_{A,2k,t}$. By Lemma 5.7, it holds that

$$\text{b-rank}(p_{A,2k}) \leq \sum_{t=1}^{2k} \text{b-rank}(q_{A,2k,t}) \leq (n-1) + 2 \sum_{t=1}^{k-1} s_t \leq n + 2(k-1).$$

□

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